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Remarks on semiclassical quantization rule for broken SUSY

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Abstract. We show that the new semiclassical quantization rule introduced for the case where supersymmetry is broken yields exact energy spectra for the isotropic harmonic oscillator in three dimensions and the Pöschl–Teller oscillator. We also show that the formula gives a good approximation for the energy spectrum for the quartic potential $V(x) = x^4 - \sqrt{2}x$.

As is well known, the WKB quantization rule yields the exact energy spectrum for the harmonic oscillator but needs the Langer replacement, $\ell(\ell+1) \to (\ell+1/2)^2$, to generate the exact spectrum for the isotropic harmonic oscillator in three dimensions. The WKB formula is also known to yield correct spectra for many of the so-called shape-invariant systems provided appropriate Langer-type modifications are applied [1]. In recent years, it has been recognized that the CBC formula, another semiclassical quantization formula proposed by Comtet *et al* [2] in relation with supersymmetric quantum mechanics, reproduces the exact spectra for those same shape-invariant systems [3]. Surprisingly, it does so without any Langer-type replacements [4]. Very recently, yet another semiclassical formula has been proposed for the cases where supersymmetry (SUSY) is broken [5]. In the present paper, we wish to show that the new quantization formula for broken SUSY can also produce the exact energy spectra for the isotropic harmonic oscillator and the Pöschl–Teller oscillator without any *ad hoc* modifications. Furthermore, we calculate the energy values for the potential $V(x) = x^4 - \sqrt{2}x$ by the broken SUSY formula and show that the results are as good as those obtained from the WKB formula.

Let us consider the bound-state problem $H\psi(q) = E\psi(q)$ with the Hamiltonian $H = p^2/(2m) + V(q)$. Let $H = \tilde{H} + \varepsilon$ and $V(q) = \tilde{V}(q) + \varepsilon$ where ε is a constant. Then we may deal with $\tilde{H}\tilde{\psi}(q) = \tilde{E}\tilde{\psi}(q)$. Suppose the potential $\tilde{V}(q)$ is given in terms of a scalar function $\phi(q)$ as

$$\tilde{V}(q) = \phi^2(q) - \sqrt{\hbar^2/(2m)}\,\phi'(q) \tag{1}$$

where $\phi'(q) = \mathrm{d}\phi/\mathrm{d}q$. The lowest eigenvalue $\tilde{E}_0$ of $\tilde{H}$ is zero when SUSY is good. In order for the ground-state eigenfunction $\psi_0(q)$ to be normalizable, the function $\phi(q)$ must have an odd number of zeros [6]. If it has an even number of zeros, SUSY is broken.

Since $E - V(q) = \tilde{E} - \tilde{V}(q)$, the WKB quantization rule for the system in question may be given by

$$\int_{x_L}^{x_R} \sqrt{2m[\tilde{E} - \tilde{V}(q)]}\,\mathrm{d}q = \left(n + \tfrac{1}{2}\right)\pi\hbar\,. \tag{2}$$

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Here x_L and x_R are the left and right turning points, respectively, which satisfy $\tilde{V}(x_L) = \tilde{V}(x_R) = \tilde{E}$, and $n \in \mathbb{N}_0$. Taking the explicit form (1) of the potential and expanding in powers of $\hbar$ (we formally treat $\phi(q)$ to be independent of $\hbar$) the integral on the left-hand side can be put into the form

$$\sqrt{2m}\int_{x_L}^{x_R}\left[\tilde{E}-\phi^2(q)+\frac{\hbar}{\sqrt{2m}}\phi'(q)\right]^{1/2}\mathrm{d}q=\int_{q_L}^{q_R}\sqrt{2m[\tilde{E}-\phi^2(q)]}\,\mathrm{d}q$$
$$+\frac{\hbar}{2}\int_{q_L}^{q_R}\frac{\phi'(q)}{\sqrt{\tilde{E}-\phi^2(q)}}\,\mathrm{d}q+O(\hbar^{3/2})\,.$$

Here we have introduced new 'turning points' q_L and q_R which are defined by the roots of $\tilde{E}=\phi^2(q)$. Inserting this result in (2) and neglecting the higher-order terms in $\hbar$ we arrive at

$$\int_{q_L}^{q_R}\sqrt{2m[\tilde{E}-\phi^2(q)]}\,\mathrm{d}q=\left(n+\frac{1}{2}-\frac{a(q_R)-a(q_L)}{2\pi}\right)\pi\hbar \tag{3}$$

where we have set $a(q)=\sin^{-1}\left(\phi(q)/\sqrt{\tilde{E}}\right)$. This quantization rule, derived earlier by Eckhardt [7] from Maslov's asymptotic analysis and by two of us [5] from Feynman's path integral, implies either the CBC formula [2]

$$\int_{q_L}^{q_R}\sqrt{2m[\tilde{E}-\phi^2(q)]}\,\mathrm{d}q=n\pi\hbar \tag{4}$$

or the broken SUSY formula [5]

$$\int_{q_L}^{q_R}\sqrt{2m[\tilde{E}-\phi^2(q)]}\,\mathrm{d}q=\left(n+\tfrac{1}{2}\right)\pi\hbar\,. \tag{5}$$

The former is valid for $\phi(q_L)=-\phi(q_R)=-\sqrt{\tilde{E}}$, and the latter for $\phi(q_L)=\phi(q_R)=\sqrt{\tilde{E}}$. If the given potential allows only two turning points, the former is valid when SUSY is good, and the latter is applicable only when SUSY is broken.

If the given potential is shape-invariant and can be given in terms of the function $\phi(q)$ of the form

$$\phi(q)=Af(q)+B/f(q) \tag{6}$$

then the eigenvalues of H can be calculated by either the CBC formula (4) or the broken SUSY formula (5). If $A/B<0$, there are two physically meaningful roots which satisfy $\phi(q_L)=-\phi(q_R)$. Then we may use the CBC formula to calculate the spectrum of $\tilde{H}$. If we choose $A/B>0$, then we have $\phi(q_L)=\phi(q_R)$ at the turning points. Hence we must use the broken SUSY formula.

The radial harmonic oscillator. The effective potential for this system is

$$V(r) = \frac{1}{2}m\omega^2 r^2 + \frac{\ell(\ell+1)\hbar^2}{2mr^2} \qquad r > 0. \tag{7}$$

The usual choice of the function $\phi(r)$ is

$$\phi(r) = \sqrt{\frac{m}{2}}\,\omega r - \frac{(\ell+1)\hbar}{\sqrt{2m}\,r}$$

which satisfies $\phi(r_L) = -\phi(r_R) = -\sqrt{\tilde{E}}$. The spectrum resulting from the CBC formula is $\tilde{E} = 2n\hbar\omega$. Obviously, $\tilde{E}_0 = 0$. Since $\varepsilon = V(r) - \tilde{V}(r) = (\ell + \frac{3}{2})\hbar\omega$, $E = \tilde{E} + \varepsilon$ gives the well known result $E = (2n + \ell + \frac{3}{2})\hbar\omega$. Alternatively, if we choose

$$\phi(r) = \sqrt{\frac{m}{2}}\,\omega r + \frac{\ell\hbar}{\sqrt{2m}\,r} \tag{8}$$

we have $\phi(r_L) = \phi(r_R) = \sqrt{\tilde{E}}$. The CBC formula is no longer applicable. We have to use (5) to calculate the spectrum which turns out to be $\tilde{E} = (2n + 2\ell + 1)\hbar\omega$. In this case, $\varepsilon = V(r) - \tilde{V}(r) = -(\ell - \frac{1}{2})\hbar\omega$. Note that $\tilde{E}_0 \neq 0$. SUSY is apparently broken. Yet, by adding $\tilde{E}$ and ε, we are able to arrive at the same exact energy spectrum.

The Pöschl–Teller oscillator. The Pöschl–Teller potential is often written in the form

$$V(x) = V_0\left[\kappa(\kappa-1)\,\mathrm{cosec}^2(\alpha x) + \lambda(\lambda-1)\,\sec^2(\alpha x)\right] \tag{9}$$

where $V_0 = \hbar^2\alpha^2/(2m)$, $\kappa > 1$, $\lambda > 1$, and $0 < \alpha x < \pi/2$. If we wish to make the ground state of $\tilde{H}$ normalizable, we must choose

$$\phi(x) = \sqrt{V_0}\,[\lambda\,\tan(\alpha x) - \kappa\,\cot(\alpha x)]$$

which satisfies $\phi(x_L) = -\phi(x_R)$. The integral of the CBC formula can be exactly computed with this function, which leads us to the spectrum $\tilde{E} = V_0[(2n + \kappa + \lambda)^2 - (\kappa + \lambda)^2]$. Evidently, $\tilde{E}$ vanishes for $n = 0$. Since $\varepsilon = V_0(\kappa + \lambda)^2$, the energy spectrum of the Pöschl–Teller oscillator is $E = V_0(2n + \kappa + \lambda)^2$. Another option is

$$\phi(x) = \sqrt{V_0}\,[\lambda\,\tan(\alpha x) + (\kappa - 1)\,\cot(\alpha x)] \tag{10}$$

which meets the condition $\phi(x_L) = \phi(x_R)$ and requires us to use the broken SUSY formula (5). The energy spectrum resulting from (5) is $\tilde{E} = V_0[(2n + \kappa + \lambda)^2 - (\kappa - \lambda - 1)^2]$. Since $\varepsilon = V_0(\kappa - \lambda - 1)^2$, we readily reach the desired spectrum. Extension of the present calculation to the case of the modified Pöschl–Teller potential is rather straightforward [8].

A quartic potential. As an example of those which are not shape-invariant, we consider the following quartic potential:

$$V(x) = a^2x^4 - \sqrt{2/m}\,a\hbar x \tag{11}$$

where a is a positive constant and $-\infty < x < \infty$. Here we choose

$$\phi(x) = ax^2 \tag{12}$$

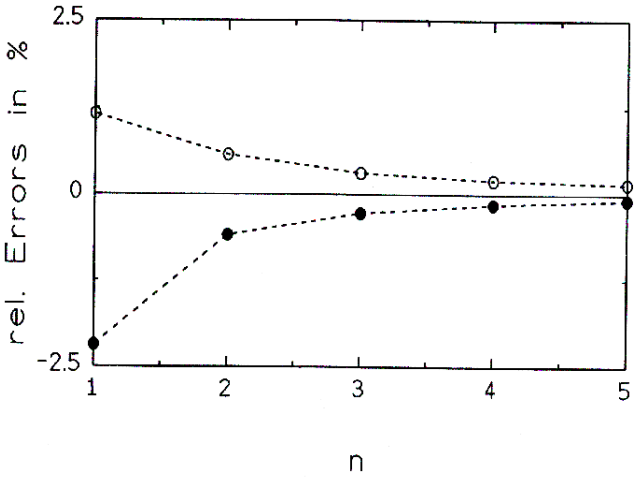

Figure 1. Relative errors for the spectrum of the quartic oscillator (11) using the WKB (●) and broken SUSY (○) approximation, respectively. Lines are drawn do guide the eye.

for which $\phi(x_L) = \phi(x_R) = \sqrt{\tilde{E}}$. Certainly, the CBC formula is not applicable. The energy spectrum calculated from the broken SUSY formula (5) can be given by a closed-form expression $E = (\hbar^2 a/m)^{2/3}[3\pi^{3/2}[\Gamma(\frac{1}{4})]^{-2}(n + \frac{1}{2})]^{4/3}$. Even though the spectrum is given in closed form, it is not an exact result. It is a semiclassical estimation. We compare numerically these values with those obtained from the WKB formula and those calculated from the Schrödinger equation. In figure 1, the deviation of the spectra approximated by the broken SUSY formula (open circles) and by the WKB formula (full circles) from the Schrödinger result is plotted for $\hbar = m = a = 1$. Note that the potential (11) obeys the scaling property $x \to \lambda x$, $a \to a/\lambda^3$, $E \to E/\lambda^2$, so the relative errors in figure 1 are independent of parameters. The broken SUSY formula (5) overestimates the energy values, while the WKB formula (2) underestimates them. We have also observed similar trends for other examples.

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[7] Eckhardt B 1986 *Phys. Lett.* **168B** 245; Also, Eckhardt has obtained the same formula (3) in a different way to that of [5]; he has never related the two possible cases (4) and (5) to unbroken and broken SUSY, respectively
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